WORKING PAPER

# When Summation and Rounding Fail to Commute

*Nine elementary questions that a fiscal product cannot leave implicit*

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**ABSTRACT**

The questions in this paper are elementary. They can be stated in one line each, and a reader who has divided integers will already suspect the answers. That is not a reason to leave them unwritten. Winstwaker is a product that must turn an integer-cent ledger into a whole-euro Dutch VAT return, and that must let software and language-model agents propose the same turn. Every wrong answer to these questions has a concrete product failure: two totals for one period, a year that is not the year, an amendment that is not the amendment, a "check" that invents tax.

This paper therefore does one thing. It states nine questions. It answers each of them by arithmetic in $\mathbb{Z}$. The long products that justify the two division lemmas sit in the appendix; the body uses those lemmas and does not reprint them. It assumes no rounding convention, no tax commentary, and no property of any implementation. The only primitives are integer addition, multiplication, and the division algorithm with divisor 100. Where a claim is used later, it is proved first.

The answers are: rounding does not commute with addition; the filed payable total is not the rounding of the cent-level total; a position that is zero in cents can file as minus one euro; a filed euro cannot be inverted; a credit of 101 cents of VAT due files as minus two euros, not minus one; four filed quarters are not the filed year; rate times base is not booked tax; a mixed remainder vector files a different net from the rounding of its cent-level net; and the amendment of a filed period is the difference of two filings, not the filing of a cent-level increment.

These facts are what Winstwaker's development is not allowed to forget. They are written here so that they do not have to be rediscovered in production.

**KEYWORDS.** integer division; directed rounding; non-commuting diagrams; value-added tax; tax amendment; product invariants.

## 1. Intention

Winstwaker keeps books in cents and must, at the end of a quarter, produce the euro amounts that belong on a Dutch VAT return. That step looks like unit conversion. It is not. Conversion that is applied to a sum is a different function from conversion applied to each summand. Anyone who has split a restaurant bill has met the phenomenon. A fiscal product that files returns, prepares amendments, and lets agents "check the VAT" meets it every day, at amounts that change what is owed.

The intention of this paper is therefore not to introduce a new theory of tax, and not to describe Winstwaker's internals. The intention is to write down, without a gap, the answers to nine questions that the product's development will otherwise answer by habit:

1. May we add after rounding?
2. May we round the payable total as if it were a box?
3. Does a zero in cents file as a zero in euros?
4. May we read cents back from a filed euro?
5. How is a small credit rounded?
6. May we add four quarterly filings to obtain a year?

7. May we replace booked tax by a statutory rate times a base?
8. Do the answers survive a vector that is not a classroom example?
9. Is a suppletie the filing of the new invoices, or the difference of two already-rounded filings?

Each question is simple. Each wrong answer is a bug that does not look like a bug: the numbers are close, the forms still add up on the page, and the disagreement is one or two euros. Winstwaker cannot be developed on that tolerance. The paper exists to remove the tolerance. If a later design choice contradicts an equality below, the equality wins.

No number in the body is imported from a codebase. Every number is computed in the section that uses it. A companion check may refuse a silent edit of those integers; it is not a source.

## 2. What is not assumed

The following are *not* taken as given.

- That "rounding to euros" is a single function. Two functions are defined, from the division algorithm, and they are shown to differ on named integers.
- That $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor x + y \rfloor$. Question 1 computes both sides.
- That the payable box may be rounded like any other box. Question 2 computes both procedures.
- That a zero cent-net files as zero. Question 3 computes a counterexample.
- That a filed euro determines the cents. Question 4 lists a fibre of cardinality 100.
- That negative amounts round toward zero. Question 5 computes both directions.
- That years are sums of filed quarters. Question 6 computes both sides.
- That booked VAT equals rate times base. Question 7 computes both sides.
- That a "realistic" mix behaves like a multiple of 100. Question 8 computes a mix that does not.
- That an amendment is the filing of the increment in cents. Question 9 computes both procedures on one already-filed period.

IEEE 754, Knuth's identities, and Dutch administrative guidance are cited only as related writing. They are not used as lemmas. If a sentence cannot be replaced by an integer calculation, it is marked as intention, not as claim.

## 3. Language

### *3.1 Integers*

Work throughout in $\mathbb{Z}$, the integers, with the usual addition and multiplication. One euro is the integer 100. An amount of $v$ cents is the integer $v$. No decimal point is used in a calculation. The symbol "€" appears only after a number that has already been obtained as a quotient; it is a label for that quotient, not a second unit.

### *3.2 The division algorithm, divisor 100*

**Lemma 1.** For every integer $v \geq 0$ there exist unique integers $q \geq 0$ and $r$ such that

$$v = 100q + r \qquad \text{and} \qquad 0 \leq r \leq 99.$$

*Existence, by explicit construction for every $v$ used later.* Given $v$, let $q$ be the largest integer with $100q \leq v$, and set $r = v - 100q$. Then $r \geq 0$ by choice of $q$, and $r \leq 99$ because if $r \geq 100$ then $100(q+1) \leq v$, contradicting maximality. Uniqueness: if $v = 100q_1 + r_1 = 100q_2 + r_2$ with both remainders in $\{0, \ldots, 99\}$, then $100(q_1 - q_2) = r_2 - r_1$, so 100 divides $r_2 - r_1$. The only multiple of 100 in $\{-99, \ldots, 99\}$ is 0, hence $r_1 = r_2$ and $q_1 = q_2$.

The pair $(q, r)$ is written $\operatorname{divrem}(v) = (q, r)$. Every quotient and remainder used later is tabulated, with the products written out, in the appendix. The body cites those lines; it does not reprint them.

### 3.3 Negative dividends

**Lemma 2.** For every integer $v < 0$ there exist unique integers $q$ and $r$ such that

$$v = 100q + r \qquad \text{and} \qquad 0 \le r \le 99.$$

*Construction.* Let $n = -v > 0$ and write $n = 100q_+ + r_+$ by Lemma 1. If $r_+ = 0$, then $v = 100(-q_+) + 0$. If $r_+ \ge 1$, then

$$v = -100q_+ - r_+ = 100(-q_+ - 1) + (100 - r_+),$$

and $1 \le 100 - r_+ \le 99$. Uniqueness is the same argument as in Lemma 1.

The only negative dividend used later is $-101$.

$$101 = 100 \cdot 1 + 1, \qquad -101 = 100 \cdot (-2) + 99,$$

because $-200 + 99 = -101$. Check: $100 \cdot (-2) = -200$, $-200 + 99 = -101$.

### 3.4 Floor and ceil, defined, not assumed

Define two functions $\mathbb{Z} \to \mathbb{Z}$ from the pair $(q, r)$ of Lemma 1 or 2.

$$\lfloor v \rfloor := q,$$

the unique quotient in $v = 100q + r$ with $0 \le r \le 99$. This is division toward $-\infty$: for $v \ge 0$ it discards $r$; for $v < 0$ it is already the more negative quotient when $r \ne 0$.

Define $\lceil v \rceil$ as the unique integer satisfying

$$\lceil v \rceil = \begin{cases} q & \text{if } r = 0, \\ q + 1 & \text{if } r \ge 1. \end{cases}$$

Then $\lceil v \rceil$ is division toward $+\infty$. Equivalently, for $v \ge 0$, $\lceil v \rceil = q$ when 100 divides $v$ and $\lceil v \rceil = q + 1$ otherwise; for $v < 0$, writing $n = -v = 100q_+ + r_+$, one has $\lceil v \rceil = -q_+$ (the remainder of $n$ is discarded, which moves toward $+\infty$).

**Lemma 3 (values used later).** Each floor is the quotient $q$ of Lemma 1 or 2. Each ceil with remainder at least 1 is that $q$ plus one. The named values — $\lfloor 199 \rfloor = 1$, $\lfloor 398 \rfloor = 3$, $\lceil 101 \rceil = 2$, $\lceil 202 \rceil = 3$, $\lfloor -101 \rfloor = -2$, and the rest — are the appendix table “Floor and ceil”. A later section that writes $\lfloor 199 \rfloor = 1$ is citing that line, not assuming a rounding convention.

### 3.5 Filing, as a pair of functions

A *due box* is an integer of cents that is filed by $\lfloor \cdot \rfloor$. An *input box* is an integer of cents that is filed by $\lceil \cdot \rceil$. If $d_1, \ldots, d_m$ are due boxes and $b$ is an input box, the *filed payable total* is defined by

$$D := \lfloor d_1 \rfloor + \cdots + \lfloor d_m \rfloor, \qquad B := \lceil b \rceil, \qquad N := D - B.$$

The *cent-level payable total* and *cent-level net* are

$$d := d_1 + \cdots + d_m, \qquad n := d - b.$$

Question 2 asks whether $D = \lfloor d \rfloor$. Question 3 asks whether $n = 0$ implies $N = 0$. Nothing else is built in.

This pair of modes is the Dutch entrepreneur-advantage convention (due never rounded up, input never rounded down). The paper does not argue for the convention. It computes its consequences.

### 3.6 Homomorphism, as an equality

A map $f : \mathbb{Z} \to \mathbb{Z}$ *preserves addition* when $f(x+y) = f(x) + f(y)$ for all $x, y$. The word "homomorphism" means that equality and nothing else. Summation of several due boxes in cents preserves addition, because addition in $\mathbb{Z}$ does. Filing does not, which is Question 1.

## 4. Question 1

*Does rounding down commute with addition? Does rounding up?*

### 4.1 Floor

Let $x = 199$ and $y = 199$. From Lemma 3,

$$\lfloor 199 \rfloor = 1, \qquad \lfloor 199 \rfloor = 1.$$

Add the filed values:

$$1 + 1 = 2.$$

Add the cents, then file:

$$199 + 199 = 398, \qquad 398 = 100 \cdot 3 + 98, \qquad \lfloor 398 \rfloor = 3.$$
$$2 \neq 3.$$

A second pair, so the disagreement is not an artefact of the remainder 99. Let $x = 101$ and $y = 199$:

$$\lfloor 101 \rfloor = 1, \qquad \lfloor 199 \rfloor = 1, \qquad 1 + 1 = 2,$$
$$101 + 199 = 300, \qquad 300 = 100 \cdot 3 + 0, \qquad \lfloor 300 \rfloor = 3.$$
$$2 \neq 3.$$

A pair that *does* commute, so the paper does not claim that every pair fails. Let $x = 100$ and $y = 200$:

$$\lfloor 100 \rfloor = 1, \qquad \lfloor 200 \rfloor = 2, \qquad 1 + 2 = 3,$$
$$100 + 200 = 300, \qquad \lfloor 300 \rfloor = 3.$$

Here both remainders are 0, and the two sides agree. Agreement on multiples of 100 is not a licence to treat agreement as the general case.

### 4.2 Ceil

Let $x = 101$ and $y = 101$.

$$\lceil 101 \rceil = 2, \qquad 2 + 2 = 4,$$
$$101 + 101 = 202, \qquad 202 = 100 \cdot 2 + 2, \qquad \lceil 202 \rceil = 2 + 1 = 3.$$
$$4 \neq 3.$$

A second pair: $x = 1$, $y = 1$.

$$\lceil 1 \rceil = 1, \qquad 1 + 1 = 2,$$
$$1 + 1 = 2, \qquad 2 = 100 \cdot 0 + 2, \qquad \lceil 2 \rceil = 1.$$
$$2 \neq 1.$$

### 4.3 Answer

No. There exist integers — exhibited above — for which

$$\lfloor x \rfloor + \lfloor y \rfloor \neq \lfloor x + y \rfloor,$$
$$\lceil x \rceil + \lceil y \rceil \neq \lceil x + y \rceil.$$

> **What Winstwaker's development must not do.** It must not add two already filed euro amounts and treat the sum as the filing of the combined cents. The disagreement on the first pair is one euro on 398 cents. That is large enough to change a return and small enough to look like a rounding trifle.

## 5. Question 2

*Is the filed payable total the rounding of the cent-level payable total?*

Let two due boxes be $d_1 = 199$ and $d_2 = 199$, and let there be no input box. Then

$$d = 199 + 199 = 398, \qquad \lfloor d \rfloor = 3$$

by §4.1. The filed payable total, by the definition in §3.5, is

$$D = \lfloor 199 \rfloor + \lfloor 199 \rfloor = 1 + 1 = 2.$$

$$D \neq \lfloor d \rfloor.$$

The same disagreement on the mixed pair of §4.1:

$$d_1 = 101,\ d_2 = 199, \qquad d = 300, \qquad \lfloor d \rfloor = 3,$$

$$D = 1 + 1 = 2, \qquad 2 \neq 3.$$

> **Answer.** No. The payable total is not a box. Filing it as if it were a box — rounding the sum of the cents — is a different function from summing the filed boxes.

> **What Winstwaker's development must not do.** It must not take a cent-level 5a, divide by 100, and print that quotient as the return's 5a. It must form 5a from the filed due boxes. Question 8 repeats the disagreement on a longer vector.

## 6. Question 3

*If the cent-level net is zero, is the filed net zero?*

Let $d_1 = 199$ (due) and $b = 199$ (input). The cent-level net is

$$n = 199 - 199 = 0.$$

The filed amounts, from Lemma 3, are

$$D = \lfloor 199 \rfloor = 1, \qquad B = \lceil 199 \rceil = 2, \qquad N = 1 - 2 = -1.$$

$$n = 0, \qquad N = -1, \qquad N \neq n/100.$$

A second witness, so the result is not tied to the remainder 99. Let $d_1 = 101$ and $b = 101$:

$$n = 0, \qquad D = \lfloor 101 \rfloor = 1, \qquad B = \lceil 101 \rceil = 2, \qquad N = 1 - 2 = -1.$$

A witness that *does* file as zero: $d_1 = 100$, $b = 100$.

$$n = 0, \qquad D = 1, \qquad B = \lceil 100 \rceil = 1, \qquad N = 0.$$

Zero files as zero when both sides are multiples of 100. That is a special case, not the definition.

> **Answer.** No. A position that is exactly settled in cents can file as minus one euro — one euro receivable — because due is floored and input is ceiled.

> **What Winstwaker's development must not do.** It must not treat "the books net to zero, so the return is zero"

> as a lemma. The return is $N$, not $n$.

## 7. Question 4

*Does a filed euro determine the cents that produced it?*

### *7.1 Two preimages of 1 under floor*

From Lemma 3,

$$\lfloor 100 \rfloor = 1, \qquad \lfloor 199 \rfloor = 1, \qquad 100 \neq 199.$$

The same filed euro, two cent amounts.

### *7.2 The whole fibre*

**Lemma 4.** For each integer $k \geq 1$,

$$\lfloor v \rfloor = k \iff v \in \{100k,\ 100k+1,\ \ldots,\ 100k+99\}.$$

That set has $99 - 0 + 1 = 100$ elements when counted from $100k$ to $100k + 99$:

$$(100k + 99) - (100k) + 1 = 100.$$

*Proof.* If $v = 100k + r$ with $0 \leq r \leq 99$, then $\lfloor v \rfloor = k$ by definition. If $\lfloor v \rfloor = k$, then $v = 100k + r$ for a unique such $r$, so $v$ lies in the set. □

For $k = 1$ the fibre is

$$\{100, 101, 102, \ldots, 199\}.$$

The four boundary checks of Lemma 3 sit on the edge of this set:

$$\lfloor 99 \rfloor = 0, \quad \lfloor 100 \rfloor = 1, \quad \lfloor 199 \rfloor = 1, \quad \lfloor 200 \rfloor = 2.$$

### *7.3 The fibre under ceil*

**Lemma 5.** For each integer $k \geq 1$,

$$\lceil v \rceil = k \iff v \in \{100(k-1)+1,\ \ldots,\ 100k\}.$$

Cardinality: $100k - (100k - 100 + 1) + 1 = 100$.

*Proof.* Write $v = 100q + r$. Then $\lceil v \rceil = k$ means either ($r = 0$ and $q = k$) or ($r \geq 1$ and $q + 1 = k$). The first case is $v = 100k$. The second is $q = k - 1$ and $1 \leq r \leq 99$, i.e. $v \in \{100(k-1)+1, \ldots, 100(k-1)+99\}$. The union is the claimed set. □

For $k = 1$:

$$\{1, 2, \ldots, 100\}.$$

Boundaries:

$$\lceil 0 \rceil = 0, \quad \lceil 1 \rceil = 1, \quad \lceil 100 \rceil = 1, \quad \lceil 101 \rceil = 2.$$

### *7.4 Answer*

No. Each filed euro $k \geq 1$ has 100 cent-preimages under floor, and 100 under ceil. An observer of the filed return cannot know whether a due box of €1 was 100 cents or 199 cents.

> **What Winstwaker's development must not do.** It must not let an agent, a control, or a year-on-year comparison treat a filed euro as if the cents were known. The cents live in the ledger. The return is a non-invertible image.

## 8. Question 5

*A credit of 101 cents of VAT due: which euro is filed? A credit of 101 cents of input VAT?*

From Lemma 2 and Lemma 3,

$$-101 = 100 \cdot (-2) + 99, \qquad \lfloor -101 \rfloor = -2.$$

The same 101 cents, rounded toward zero, would have been $-1$, because $101 = 100 \cdot 1 + 1$ and the sign would have been kept on the quotient 1. Toward-zero is not the function defined in §3.4. It is mentioned only to name the integer that must *not* be used.

Input side:

$$101 = 100 \cdot 1 + 1, \qquad \lceil -101 \rceil = -1.$$

Together, a due credit of 101 and an input credit of 101 file as

$$N = \lfloor -101 \rfloor - \lceil -101 \rceil = -2 - (-1) = -1.$$

The cent-level net is $-101 - (-101) = 0$. Again $n = 0$ and $N = -1$, the signed form of Question 3.

> **Answer.** A 101-cent due credit files as $-€2$. A 101-cent input credit files as $-€1$. Directed rounding does not round toward zero.

> **What Winstwaker's development must not do.** It must not implement negative filing as "drop the remainder and keep the sign." That is toward-zero. Due credits become one euro too small in magnitude; the return is then wrong on every credit note whose remainder is not zero.

## 9. Question 6

*May four quarterly filed returns be added to obtain the filed year?*

Let each of four quarters contain a single due remainder of 199 cents and nothing else.

One quarter, from Lemma 3:

$$\lfloor 199 \rfloor = 1.$$

Four filed quarters, added as euros:

$$1 + 1 + 1 + 1 = 4.$$

Four quarters, added as cents:

$$199 + 199 = 398, \qquad 398 + 199 = 597, \qquad 597 + 199 = 796.$$

Check the last sum: $597 + 200 = 797$, so $597 + 199 = 796$. Also $4 \cdot 199 = 4 \cdot 200 - 4 = 800 - 4 = 796$.

The year, filed once:

$$796 = 100 \cdot 7 + 96, \qquad \lfloor 796 \rfloor = 7.$$
$$4 \neq 7.$$

The disagreement is three euros on a year whose only VAT is four remainders of 199 cents. No turnover was invented. The only operation that produced the 4 was the one Question 1 already forbade, applied three times.

A second year, with a remainder that *does* add: four quarters of 100 cents.

$$\lfloor 100 \rfloor = 1, \qquad 1 + 1 + 1 + 1 = 4,$$
$$4 \cdot 100 = 400, \qquad \lfloor 400 \rfloor = 4.$$

The sides agree because each remainder is 0. Agreement on this year does not license adding filed quarters in general.

> **Answer.** No. The filed year is $\lfloor d_{\text{year}} \rfloor$ (or the analogue of $D$ if several boxes are present). It is not the sum of the four filed quarterly $D$'s.

> **What Winstwaker's development must not do.** It must not produce an annual picture, a dashboard total, or an agent summary by adding four quarterly PDFs. It must fold the year's cents and file once.

## 10. Question 7

*If the books record 2 000 cents of VAT on a 10 000-cent base at a rate of 21%, does "21% of the base" recover the booked tax?*

Twenty-one percent is the integer ratio $21/100$, or $2100$ basis points on $10000$. The integer

$$\frac{10\,000 \cdot 2\,100}{10\,000}$$

is evaluated as a product and a quotient, not as a decimal:

$$10\,000 \cdot 2\,100 = 21\,000\,000,$$

because $10\,000 \cdot 2\,000 = 20\,000\,000$ and $10\,000 \cdot 100 = 1\,000\,000$, and $20\,000\,000 + 1\,000\,000 = 21\,000\,000$. Then

$$21\,000\,000/10\,000 = 2\,100,$$

because $2\,100 \cdot 10\,000 = 21\,000\,000$. The books, by the question's hypothesis, record $2\,000$.

$$2\,100 \neq 2\,000.$$

The difference is 100 cents, which files as one euro.

> **Answer.** No. Rate times base is one integer. Booked tax is another. A return that copies the books reports $2000$. A return that recomputes the rate reports $2100$. Those are two functions. Winstwaker's development has to choose the first: the return is a projection of what was booked, not a second calculation of what "should" have been booked. A later control may *compare* the two integers (Question 7 names them). It may not silently replace one by the other.

> **What Winstwaker's development must not do.** It must not let an agent "verify" box 1a by computing 0.21 times turnover and calling a disagreement a rounding error. The disagreement of 100 cents is exact.

## 11. Question 8

*Do Questions 1–2 survive a vector that is not a multiple of 100 in every coordinate?*

Let the due boxes be $1841$, $500$, and $467$ cents, and let the input box be $934$ cents.

### *11.1 Cent-level sums*

$$1841 + 500 = 2341,$$

because $1800 + 500 = 2300$ and $2300 + 41 = 2341$.

$$2341 + 467 = 2808,$$

because $2341 + 400 = 2741$ and $2741 + 67 = 2808$. Check against §3.2: $2808 = 100 \cdot 28 + 8$.

$$2808 - 934 = 1874,$$

because $2808 - 900 = 1908$ and $1908 - 34 = 1874$. Check: $1874 = 100 \cdot 18 + 74$.

So $d = 2808$ and $n = 1874$.

### *11.2 Filing each box*

From Lemma 3:

$$\lfloor 1841 \rfloor = 18, \qquad \lfloor 500 \rfloor = 5, \qquad \lfloor 467 \rfloor = 4, \qquad \lceil 934 \rceil = 10.$$

### *11.3 Filed totals*

$$D = 18 + 5 + 4.$$

$18 + 5 = 23$, $23 + 4 = 27$. So $D = 27$.

$$N = 27 - 10 = 17.$$

### *11.4 The two forbidden shortcuts*

$$\lfloor d \rfloor = \lfloor 2808 \rfloor = 28, \qquad 28 \neq 27,$$
$$\lfloor n \rfloor = \lfloor 1874 \rfloor = 18, \qquad 18 \neq 17.$$

Question 2, on this vector: the filed payable is 27, not 28. The filed net is 17, not 18.

### *11.5 Answer*

Yes, the disagreements survive. The filed net is €17. It is not the rounding of the cent-level net.

> **What Winstwaker's development must not do.** It must not treat a "messy" period as a reason to switch to rounding the totals. The messy period is the ordinary case. Question 8 is the ordinary case written out.

## 12. Question 9

*Is the amendment of a filed period the filing of the new invoices, or the difference between two already-rounded filings?*

### *12.1 Two procedures on one period*

A period has already been filed. Later bookings change the cents. Write $d$ for the due-box cents that were filed, and $\delta$ for the increment in the same box — the late invoice, the backlog booking. Two procedures present themselves.

**Procedure $S_R$** (difference of two filings):

$$S_R := \lfloor d + \delta \rfloor - \lfloor d \rfloor.$$

File the old snapshot. File the new snapshot. Subtract. That difference is the amendment.

**Procedure $S_C$** (file the increment):

$$S_C := \lfloor \delta \rfloor.$$

Take the new invoices as cents, file that vector once, and call the result the correction.

Question 6 asked whether four filed periods sum to the filing of their union. Both sides of Question 6 start from unfiled cents. Question 9 starts from a period that has already been filed, and asks whether the correction is the filing of the increment. A reader who accepted Question 6 can still believe $S_C$: the new invoices look like a fresh return of their own.

### *12.2 The increment identity*

**Lemma 6.** Write $d = 100q + r$ and $\delta = 100s + t$ with $0 \le r, t \le 99$. Then

$$\lfloor d+\delta \rfloor - \lfloor d \rfloor = \begin{cases} \lfloor \delta \rfloor & \text{if } r+t < 100, \\ \lfloor \delta \rfloor + 1 & \text{if } r+t \ge 100. \end{cases}$$

*Proof.* $d + \delta = 100(q+s) + (r+t)$. If $r + t < 100$, the remainder is already in range, so $\lfloor d+\delta \rfloor = q + s$ and the left-hand side is $s = \lfloor \delta \rfloor$. If $r + t \ge 100$, write $r + t = 100 + u$ with $0 \le u \le 98$, so $\lfloor d+\delta \rfloor = q + s + 1$ and the left-hand side is $s + 1$.

Hence $S_R = S_C$ if and only if the two remainders do not overflow. That is not the general case, and it is not the backlog case.

### *12.3 Due box, same rubric grows*

Let $d = 199$ and $\delta = 199$. From Lemma 3, $\lfloor 199 \rfloor = 1$. The new snapshot is $199 + 199 = 398$, and $\lfloor 398 \rfloor = 3$.

$$S_R = 3 - 1 = 2, \qquad S_C = \lfloor 199 \rfloor = 1.$$

$$2 \ne 1.$$

The remainders are $99 + 99 = 198$, and $198 \ge 100$, so Lemma 6 already predicted the extra euro.

A third procedure also presents itself in backlog work: treat the amendment as the new filing itself,

$$S_N := \lfloor d+\delta \rfloor = 3.$$

That re-declares the original euro. It is not an increment at all.

### *12.4 Input box*

Let $b = 101$ and $\varepsilon = 101$. Then $\lceil 101 \rceil = 2$ and $\lceil 202 \rceil = 3$.

$$S_R = 3 - 2 = 1, \qquad S_C = \lceil 101 \rceil = 2.$$

$$1 \ne 2.$$

Filing the increment of input overstates the correction by one euro. The same overflow that added a euro on the due side removes a euro on the input side, because the ceil of a sum is not the sum of ceils (Question 1).

### *12.5 Why this is not Question 6*

Question 6 compared $\sum_i \mathrm{file}(Q_i)$ with $\mathrm{file}(\sum_i Q_i)$ across four periods. Both sides start from unfiled cents. Question 9 starts from a period that has already been filed. The old filing is a photograph; the later bookings are the film. The amendment is the difference of two photographs, not a third photograph of the new frames alone.

A product that answers Question 6 correctly can still compute a suppletie by rounding the cent-level delta. That is $S_C$. It produces a different integer from $S_R$ on the same named pair $(199, 199)$.

### *12.6 Answer*

The amendment is $S_R$: file both snapshots, subtract. It is not $S_C$: file the increment. It is not $S_N$: send the new filing as the correction.

> **What Winstwaker's development must not do.** It must not compute a suppletie by applying $\lfloor \cdot \rfloor$ or $\lceil \cdot \rceil$ to the difference of two cent-vectors. It files the old declaration, files the new declaration, and subtracts those euros. "Previously declared" is the already-filed photograph. The correction is not a re-file of the backlog invoices.

## 13. What the nine answers require of Winstwaker

The questions are simple. The product consequences are not optional.

1. **Compute in cents.** Every intermediate total in a prepare, an amendment, a dashboard, or an agent draft is an integer of cents until the last step.
2. **File once.** $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are applied to each box, then $D$ and $N$ are formed from those quotients. They are not applied to $d$ or $n$.
3. **Do not add filings of different periods.** Four quarterly filings are not a year. The year is computed from the year's cents.
4. **Do not invert a filing.** A control that sees only euros does not know the cents (Lemma 4, Lemma 5). The ledger is the preimage.
5. **Do not recompute tax from a rate.** Booked tax is the input to filing. Rate times base is a comparison, not a substitute.
6. **Do not round credits toward zero.** $\lfloor -101 \rfloor = -2$.
7. **Do not file the increment.** A suppletie is $\mathrm{file}(\mathrm{current}) - \mathrm{file}(\mathrm{previous})$, both already rounded per box. It is not $\mathrm{file}(\mathrm{current} - \mathrm{previous})$ in cents, and it is not the new filing sent as the correction.

These seven sentences are the development rule. The paper is the proof that they are not style. Each is the negation of a procedure that produces a different integer from the one the return, or the amendment, must show.

## 14. Related writing

Pacioli (1494) treats the journal as a conservation law: every fact is written twice. Ellerman (1985) recasts that conservation as a linear condition on a module of T-accounts. The present paper is about a second map, out of that module, which is not linear. The non-linearity is not a new theorem about floor and ceil — it is the content of Question 1, written as school arithmetic because school arithmetic is what the product will actually run.

IEEE 754-2019 names directed rounding. Knuth (1997, Vol. 1) records floor and ceiling identities. Neither text is used as a lemma here. The identities that the paper needs are Lemma 1 through Lemma 6, each with the integers that Winstwaker's filing will meet.

Dutch administrative notes describe filing in whole euros and the entrepreneur-advantage pair of modes. They do not write $D \neq \lfloor d \rfloor$. That missing equality is why a product can implement the notes and still compute the wrong 5a.

## 15. Conclusion

Nine questions, all of a kind that is easy to skip. The answers, all computed:

| | Question | Sides | Integers |
|---|---|---|---|
| 1 | Floor/ceil commute with $+$? | no | $1+1=2\neq 3$, $2+2=4\neq 3$ |
| 2 | $D = \lfloor d \rfloor$? | no | $2 \neq 3$ |
| 3 | $n = 0 \Rightarrow N = 0$? | no | $N = -1$ |
| 4 | Invertible? | no | fibre size 100 |
| 5 | Credit 101 due? | $-2$ | $-200 + 99 = -101$ |
| 6 | Year $=$ sum of quarters? | no | $4 \neq 7$ |
| 7 | Rate $\times$ base $=$ booked? | no | $2100 \neq 2000$ |
| 8 | Ordinary mix? | $N = 17$ | $28 \neq 27$, $18 \neq 17$ |

| | Question | Sides | Integers |
|---|---|---|---|
| 9 | Amendment = file the increment? | no | $S_R = 2 \neq 1 = S_C$ |

Winstwaker's development needs these answers in writing because the questions look too simple to deserve writing. The cost of leaving them implicit is a second total for the same period. The paper is the first total, fully expanded, so that the second one can be refused.

## References

Ellerman, D. P. (1985). The mathematics of double entry bookkeeping. *Mathematics Magazine*, 58(4), 226–233.

IEEE Computer Society. (2019). *IEEE Standard for Floating-Point Arithmetic* (IEEE 754-2019).

Knuth, D. E. (1997). *The Art of Computer Programming*, Vol. 1: *Fundamental Algorithms* (3rd ed.). Addison-Wesley.

Pacioli, L. (1494). *Summa de arithmetica, geometria, proportioni et proportionalita*. Venice.

## Appendix. Calculation ledger

Every product, remainder, floor and ceil used in the body, in one place. Each line is recomputed from the two factors on the right. Section 3 cites these tables and does not reprint them.

### *Division with remainder, divisor 100*

| | |
|---|---|
| $99 = 100 \cdot 0 + 99$ | $100 = 100 \cdot 1 + 0$ |
| $101 = 100 \cdot 1 + 1$ | $199 = 100 \cdot 1 + 99$ |
| $200 = 100 \cdot 2 + 0$ | $202 = 100 \cdot 2 + 2$ |
| $398 = 100 \cdot 3 + 98$ | $467 = 100 \cdot 4 + 67$ |
| $500 = 100 \cdot 5 + 0$ | $796 = 100 \cdot 7 + 96$ |
| $934 = 100 \cdot 9 + 34$ | $1841 = 100 \cdot 18 + 41$ |
| $1874 = 100 \cdot 18 + 74$ | $2000 = 100 \cdot 20 + 0$ |
| $2100 = 100 \cdot 21 + 0$ | $2808 = 100 \cdot 28 + 8$ |
| $-101 = 100 \cdot (-2) + 99$ | |

### *Products*

| Identity | Expansion |
|---|---|
| $100 \cdot 1 = 100$ | $100$ |
| $100 \cdot 2 = 200$ | $100 + 100$ |
| $100 \cdot 3 = 300$ | $200 + 100$; $300 + 98 = 398$ |
| $100 \cdot 4 = 400$ | $300 + 100$; $400 + 67 = 467$ |
| $100 \cdot 5 = 500$ | $400 + 100$ |
| $100 \cdot 7 = 700$ | $500 + 200$; $700 + 96 = 796$ |
| $100 \cdot 9 = 900$ | $500 + 400$; $900 + 34 = 934$ |
| $100 \cdot 18 = 1800$ | $10 \cdot 180 = 1800$; $1800 + 41 = 1841$; $1800 + 74 = 1874$ |
| $100 \cdot 20 = 2000$ | $2 \cdot 1000$ |
| $100 \cdot 21 = 2100$ | $2000 + 100$ |

| Identity | Expansion |
|---|---|
| $100 \cdot 28 = 2800$ | $1800 + 1000$; $2800 + 8 = 2808$ |
| $100 \cdot (-2) = -200$ | $-(100 \cdot 2)$ |
| $199 + 199 = 398$ | $200 + 198 = 398$ |
| $101 + 199 = 300$ | $100 + 200$ |
| $101 + 101 = 202$ | $200 + 2$ |
| $4 \cdot 199 = 796$ | $800 - 4$ |
| $199 + 199 + 199 + 199 = 796$ | $398 + 398 = 796$ |
| $1841 + 500 = 2341$ | $2300 + 41$ |
| $2341 + 467 = 2808$ | $2741 + 67$ |
| $2808 - 934 = 1874$ | $1908 - 34$ |
| $18 + 5 + 4 = 27$ | $23 + 4$ |
| $27 - 10 = 17$ | $17$ |
| $10\,000 \cdot 2\,100 = 21\,000\,000$ | $20\,000\,000 + 1\,000\,000$ |
| $21\,000\,000/10\,000 = 2\,100$ | inverse of the previous line |
| $-200 + 99 = -101$ | $-101$ |
| $-2 - (-1) = -1$ | $-2 + 1$ |
| $3 - 1 = 2$ | Question 9, $S_R$ due |
| $3 - 2 = 1$ | Question 9, $S_R$ input |
| $99 + 99 = 198$ | remainder overflow, Lemma 6 |

### *Floor and ceil*

| Floor | | Ceil | |
|---|---|---|---|
| $\lfloor 99 \rfloor$ | 0 | $\lceil 0 \rceil$ | 0 |
| $\lfloor 100 \rfloor$ | 1 | $\lceil 1 \rceil$ | 1 |
| $\lfloor 101 \rfloor$ | 1 | $\lceil 100 \rceil$ | 1 |
| $\lfloor 199 \rfloor$ | 1 | $\lceil 101 \rceil$ | 2 |
| $\lfloor 200 \rfloor$ | 2 | $\lceil 199 \rceil$ | 2 |
| $\lfloor 202 \rfloor$ | 2 | $\lceil 202 \rceil$ | 3 |
| $\lfloor 398 \rfloor$ | 3 | $\lceil 934 \rceil$ | 10 |
| $\lfloor 467 \rfloor$ | 4 | $\lceil -101 \rceil$ | $-1$ |
| $\lfloor 500 \rfloor$ | 5 | | |
| $\lfloor 796 \rfloor$ | 7 | | |
| $\lfloor 1841 \rfloor$ | 18 | | |
| $\lfloor 1874 \rfloor$ | 18 | | |
| $\lfloor 2808 \rfloor$ | 28 | | |
| $\lfloor -101 \rfloor$ | $-2$ | | |

Each left-hand side is the $q$ of Lemma 1 or 2. Each ceil with remainder at least 1 is that $q$ plus one: $\lceil 101 \rceil = 1 + 1 = 2$, $\lceil 199 \rceil = 1 + 1 = 2$, $\lceil 202 \rceil = 2 + 1 = 3$, $\lceil 934 \rceil = 9 + 1 = 10$. For $-101$, Lemma 2 gave $q = -2$, so $\lfloor -101 \rfloor = -2$; and $101 = 100 \cdot 1 + 1$, so $\lceil -101 \rceil = -1$.