Allocating Cents Without Losing the Total

Conservation, quota and the limits of largest-remainder allocation

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5 September 2026

ABSTRACT

Splitting an integer-cent amount in proportion to weights creates fractional claims on indivisible units. This note derives largest-remainder allocation using only integer division. It proves exact conservation, quota compliance and minimum absolute and squared deviation among quota allocations. A small counterexample shows why increasing the total can still reduce one recipient's allocation. The method is classical; the contribution of this note is a compact specification and proof suitable for discounts, shared costs and financial calculators.

Expository note: elementary results with explicit proofs. No claim of mathematical novelty or measured product performance.

1. The unit that cannot be divided

Let N be a nonnegative integer number of cents. There are n recipients with nonnegative integer weights w_1,...,w_n, not all zero. Write W for their sum. The ideal share of recipient i is q_i = N w_i / W. Ideal shares add to N, but they need not be integers. A valid allocation a must contain nonnegative integers and satisfy the exact conservation equation.

$$\sum_{i=1}^{n} a_i = N.$$

The weights express an input policy. They may be quantities, agreed proportions or another nonnegative basis. The following mathematics cannot decide which policy is legally or commercially appropriate. It specifies what happens after N and the weights have been chosen.

2. Construct the allocation using integer arithmetic

Divide each integer product N w_i by W. Let f_i be the quotient and r_i the remainder. No binary floating-point representation is required.

$$N\, w_i = W\, f_i + r_i,\ 0 \le r_i < W.$$

Define R = N − sum_i f_i. Give one additional cent to the R largest remainders and none to the others. Break ties by a fixed recipient order that is independent of arrival order. The resulting allocation is a_i = f_i + e_i, where e_i is one for a selected recipient and zero otherwise.

Proposition 1. R is an integer with 0 ≤ R < n, and the construction allocates exactly N cents.

Proof. Sum the division equations. Since sum_i w_i = W, they give WN = W sum_i f_i + sum_i r_i. Therefore WR = sum_i r_i. Each remainder is nonnegative and less than W, so 0 ≤ WR < nW. Division by the positive W gives 0 ≤ R < n. Selecting exactly R recipients adds exactly R to the quotient total, yielding sum_i a_i = sum_i f_i + R = N.

Proposition 2. Every allocation lies between the floor and ceiling of its ideal share. In particular, $|a_i - q_i| < 1$ for each recipient.

Proof. Write $t_i = r_i/W$, so $q_i = f_i + t_i$ and $0 \le t_i < 1$. If $t_i = 0$, that recipient is not selected: when $R > 0$, the number of positive t_i is strictly greater than their integer sum R, so at least R positive remainders outrank every zero. Thus an integral share stays exact. For $0 < t_i < 1$, the construction chooses either f_i or $f_i + 1$, whose deviations are t_i and $1 - t_i$. Both are strictly less than one.

3. What the construction minimises

Consider all quota allocations with the same total: each recipient receives its floor or ceiling, integral ideal shares stay fixed, and exactly R nonintegral shares are rounded upward. The next result concerns this explicitly defined class. It does not claim that every possible notion of fairness is equivalent to absolute error.

Proposition 3. Selecting the R largest fractional parts minimises both total absolute deviation and total squared deviation among quota allocations.

Proof. For a fractional part t, choosing the floor has absolute cost t and choosing the ceiling has cost $1 - t$. The additional cost of rounding upward is therefore $1 - 2t$. The sum of the floor costs is fixed, so minimising total cost means choosing R of these additional costs with the smallest sum. These are exactly the R largest t. For squared deviation, the additional cost is $(1 - t)^2 - t^2 = 1 - 2t$, giving the same ordering. If a proposed selection contains $t_i < t_j$ while j is unselected, swapping i for j reduces either cost by $2(t_j - t_i)$. Repeating such exchanges reaches a largest-remainder selection.

Equal remainders give equal costs, so minimisation alone does not choose between tied recipients. A fixed tie-break makes the output deterministic. It does not make the tied alternatives mathematically unequal or prove a legal entitlement to the extra cent.

4. An example that can be checked by hand

Allocate N = 1,001 cents with weights (5,3,2), so W = 10. The quotient total is 500 + 300 + 200 = 1,000 and R = 1. The first recipient has the largest remainder and receives the extra cent.

Recipient	N × weight	Quotient	Remainder	Allocation
A	5,005	500	5	501
B	3,003	300	3	300
C	2,002	200	2	200

$$501 + 300 + 200 = 1{,}001.$$

The ideal shares are 500.5, 300.3 and 200.2 cents. The absolute deviations are 0.5, 0.3 and 0.2 cents, summing to one cent. Giving the extra cent to B instead would give deviations 0.5, 0.7 and 0.2, summing to 1.4 cents. These are rational comparisons of the specified objective, not rounded monetary outputs.

5. A larger total can reduce one allocation

Proposition 4. Largest-remainder allocation is not monotone in the total N for every recipient, even with fixed positive weights and no remainder ties.

> *Proof.* Use weights (5,3,1), with $W = 9$. At $N = 4$ the quotient vector is (2,1,0), the remainders are (2,3,4), and $R = 1$. The allocation is (2,1,1). At $N = 5$ the same quotient vector has remainders (7,6,5) and $R = 2$. The allocation is (3,2,0). The total increased by one, but the third recipient lost its cent. There are no ties in either ranking.

This is the classical house-monotonicity failure associated with the largest-remainder method in apportionment. Replacing seats with cents does not remove the arithmetic. Consequently, a system must not infer recipient-wise monotonicity from exact conservation and small rounding error.

6. Scope for a financial calculator

The specification requires $N \geq 0$, nonnegative weights and $W > 0$. A zero total allocates zero to every recipient; zero-weight recipients receive zero. A negative correction can be given a separate sign-consistent policy, but it is not covered by silently applying the nonnegative proof to negative remainders.

Independent allocations of a transaction and of a later change need not equal a fresh allocation of the revised total. If a product requires that equality, it must retain and compare the original allocation or choose another explicitly justified policy. That choice is separate from the one-cent quota bound.

The allocation rule is the established Hamilton or largest-remainder method. This note claims neither its invention nor that it is universally preferable. It supplies the invariants and a concrete limitation that a Rekenbox-style calculator can expose and test.

References

U.S. Census Bureau. Historical Perspective: the Hamilton method and its house-monotonicity limitation.

Kylian de Groot. When Summation and Rounding Fail to Commute. Working paper, 2025.

Companion verification. Download the finite checks (Python 3). Exact integer and rational checks supplement the proofs; they do not establish universal claims beyond the checked domains.